Gradated Lines Information Sheet

Math in Fabric Design
William J. Jones, Designer Software LLC

Jhane Barnes posed the following question to Dr. William J. Jones: “Can the computer give me a color sequence for weaving that’s all red threads at one end, all gold threads at the other end, with successive gradations in between? And it has to work with any number of threads, from just a few to thousands.”

Dr. Jones went to work. Here’s how such a request is answered, illustrating how mathematics comes in where it’s least expected.

The process of color blending is like solving a little puzzle. A statement in words must evolve into a sequence of basic arithmetic operations that a computer will understand.

Suppose we have a warp of 300 threads, and we want to do such a blend. We want to create a gradation of red (R) and gold (G) threads. A “natural”idea is to divide the warp into equal sized blocks so that the first block is all red except for one gold thread at the end, each successive block has one less red and one more gold, and the last block comes out right: one red and the rest gold. How do we figure the block size and number of blocks to make this happen?

You can often solve these puzzles by looking at the simplest examples until you see a pattern. In the following, R = red and G = gold. The space between blocks is for clarity; in the fabric we’d expect all threads to be evenly spaced. We’ll try blocks of 2, 3, 4, . . . until we see what’s happening.

RG(1 block, 2 threads; one block meets the conditions of first and last block.)
RRG RGG (2 blocks $\times$ 3 threads = 6 threads)
RRRG RRGG RGGG (3 blocks $\times$ 4 threads = 12 threads)
RRRRG RRRGG RRGGG RGGGG (4 blocks $\times$ 5 threads = 20 threads)
RRRRRG RRRRGG RRRGGG RRGGGG RGGGGG (5 blocks $\times$ 6 threads = 30 threads)

The pattern is taking shape: if the block size is N threads, then there are N-1 blocks (or one less block than threads) in a full blend sequence for that size, so the sequence has $N \times (N-1)$ threads altogether. Our task is the reverse of this formula. If there are W threads, what does N have to be to make $N \times (N-1)$ as close as possible to W?

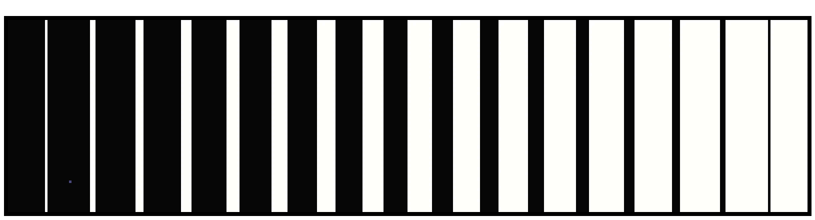
If W is a number in the series 2, 6, 12, 20, 30 . . . we can fit a sequence from the above family exactly, but this won’t usually be the case. We need to find the value of N that makes $N \times (N-1)$ as close as possible to W, restricted by the fact that N has to be an integer because it represents a number of threads.. So our equation is:

> $\mathbf{N \times (N-1) = W}$ where W is known, say 300 threads, and we have to solve
> $\mathbf{N^2 - N - 300 = 0}$ looking for the closest positive integral N to the positive root.

We can solve this using the quadratic formula for roots of $\mathbf{ax^2 + bx + c = 0}$:

$$\mathbf{x = (- b \pm \sqrt{ b^2 - 4ac }) \div 2a} \quad (a = 1,\ b = -1,\ c = -300)$$

In this case the positive root is $(1 + \sqrt{1201}) \div 2$, or 17.83. The closest integral N to this value is 18, which would make a sequence of $N \times (N-1)$ or $18 \times 17 = 306$ threads. Since **W** is 300, we can simply construct the 306-thread sequence and lop off three threads from each end. Here’s how the 300-thread blend looks:



If the fractional part had been less than 0.5, we would round down and add threads at each end the same color as the end threads. In the case we are looking at, you can see that the end stripes, black at the left and white at the right, are a little narrower than their neighbors of the same color, because 17-thread stripes were chopped to 14 to make the blend fit in 300 threads. What we are doing with shortage or excess is adjusting the size of the thickest stripes, which happen to be at the ends of the pattern. This “lumped” trimming or padding isn’t bad when the quadratic root is near an integer, but can be more visible when the fraction in the root is near one half; we’ll mention a better technique a little later.

We’ve solved the problem as given but a similar one presents itself: if we were to repeat the pattern by abutting a copy of it on the right, there would be an abrupt change from wide white stripes to wide black stripes. Can we construct a blend that goes black to white to black gradually so it repeats smoothly?

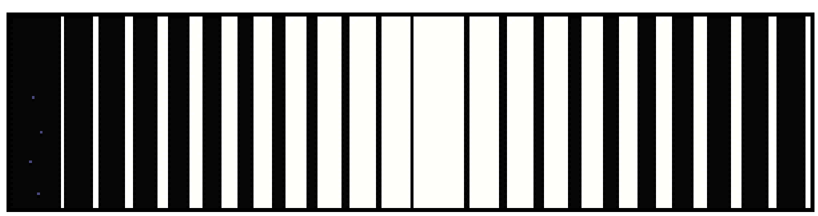
Sure we can; it’s just a new puzzle. Looking at the alternate color stripe widths, we want something like:

Black	6		5		4		3		2		1		2		3		4		5	
White		1		2		3		4		5		6		5		4		3		2

which makes a smooth up/down sequence when repeated. The block size N (one Black and one White stripe) in this example is 7, and if we try a few values of N we see that there are always $2 \times N - 4$ blocks (10 in this example) that need to add up to a number close to the number of threads W. So our equation is

$$\mathbf{N \times (2 \times N - 4) = W} \quad \text{or} \quad \mathbf{2N^2 - 4N - W = 0,}$$

which we can solve with the quadratic formula as before. For W = 300, this gives us N = 13.28, so we will try N = 13. This time $N \times (2 \times N - 4) = 286$, so we will have to do some padding to the tune of 14 threads. But now the thickest stripes aren’t at the ends as they were before; the thickest white stripe is in the middle, so the “lumped” adjustment requires finding the largest black and white stripes and adding 7 threads to each. Here is what the resulting blend would look like:



When this pattern is repeated, the stripe widths in both colors gradate continuously. The 7-thread thickening of the widest stripes is visible when you look for it, but isn’t bad. What’s important visually is balance: think about connecting the two ends of the pattern to form a “bracelet” (which is like putting it in repeat). In the bracelet, the white and black parts would be identical but with widest stripes halfway around from each other and both color stripes symmetrically arranged about the widest ones.

The better technique alluded to above is to distribute the excess or shortage as evenly as possible over all the stripes (keeping that balance!), and this has been done, too, but that’s another story. There are many variations possible by adding more rules, like specifying a maximum stripe width or requiring that black stripes vary while white stripes stay the same width. These variations may drastically change how the puzzle is solved as well as the resulting look. Then you might talk about blending in a third color, and it’s a whole new ball game!

WJJ - Designer Software LLC September 4, 1999